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On a class of K-contact manifolds

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Abstract. The object of this paper is to study K-contact manifolds with quasi-conformal curvature tensor. We characterised K-contact manifolds satisfying certain curvature conditions on the quasi-conformal curvature tensor.

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§1. Introduction

Let (M^n, g) , $(n = 2m+1)$ be a contact Riemannian manifold with contact form η , associated vector field ξ , $(1, 1)$ - tensor field ϕ and associated Riemannian metric g . If ξ is a Killing vector field, then M^n is called a K-contact manifold ([2], [15]). K-contact manifolds have been studied by several authors such as S. Tanno [18], [19], [20], S. Sasaki [16], D. E. Blair [2], Y. Hatakeyama, Y. Ogawa and S. Tanno [11], M. C. Chaki and D. Ghosh [5], U. C. De and S. Biswas [7] and many others.

The notion of the quasi-conformal curvature tensor was given by Yano and Sawaki [22]. According to them a quasi-conformal curvature tensor was given by

$$\begin{aligned}
 (1.1) \quad \tilde{C}(X, Y)Z &= aR(X, Y)Z \\
 &+ b[S(Y, Z)X - S(X, Z)Y \\
 &+ g(Y, Z)QX - g(X, Z)QY] \\
 &- \frac{r}{n} \left(\frac{a}{n-1} + 2b \right) [g(Y, Z)X - g(X, Z)Y],
 \end{aligned}$$

where a and b are constants and R , S , Q and r are the Riemannian curvature tensor of type $(1, 3)$, the Ricci tensor of type $(0, 2)$, the Ricci operator defined by $S(X, Y) = g(QX, Y)$ and scalar curvature of the manifold respectively. If $a = 1$ and $b = -\frac{1}{n-2}$, then (1.1) takes the form

$$\begin{aligned}
 (1.2) \quad \tilde{C}(X, Y)Z &= R(X, Y)Z \\
 &\quad - \frac{1}{n-2}[S(Y, Z)X - S(X, Z)Y \\
 &\quad + g(Y, Z)QX - g(X, Z)QY] \\
 &\quad + \frac{r}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y] \\
 &= C(X, Y)Z,
 \end{aligned}$$

where $C(X, Y)Z$ is the conformal curvature tensor ([10]). Thus the conformal curvature tensor C is a particular case of the tensor \tilde{C} . A manifold (M^n, g) , ($n > 3$) is called quasi-conformally flat if the quasi-conformal curvature tensor $\tilde{C} = 0$. It is known ([1]) that the quasi-conformally flat manifold is either conformally flat if $a \neq 0$ or Einstein if $a \neq 0, b \neq 0$. Since they give no restriction for manifold if $a = 0$ and $b = 0$, it is essential for us to consider the case of $a \neq 0$ or $b \neq 0$. Recently De and Matsuyama [6] studied quasi-conformally flat manifold. Also quasi-conformal curvature tensor have been studied by Özgür and De [13], De and Gazi [8] and many others. A Riemannian manifold satisfying $R(X, Y).R = 0$ is called semisymmetric ([17]), where $R(X, Y)$ denotes the derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y . In an analogous way we define quasi-conformally semisymmetric manifold. A K-contact manifold is said to be quasi-conformally semisymmetric if $R(X, Y).\tilde{C} = 0$, where \tilde{C} is the quasi-conformal curvature tensor.

The paper organised as follows:

After preliminaries in section 3, we first prove that a quasi-conformally flat K-contact manifold is an η -Einstein manifold. As a consequence of this we obtain that a quasi-conformally flat K-contact manifold is Sasakian. Section 4 deals with the study of a K-contact manifold satisfying $\text{div}\tilde{C} = 0$ and we prove that such a K-contact manifold is also Sasakian. Section 5 is devoted to the study of a K-contact Einstein (or η -Einstein) quasi-conformally semisymmetric manifold. In section 6 we prove that a ξ -quasi-conformally flat K-contact manifold is an η -Einstein manifold. Finally some applications are given.

§2. Preliminaries

By a contact manifold we mean an $n = (2m + 1)$ - dimensional differentiable manifold M^n which carries a global 1-form η , there exists a unique vector field

ξ , called the characteristic vector field such that, $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$. A Riemannian metric g on M^n is said to be an associated metric if there exists a $(1, 1)$ tensor field ϕ such that

$$(2.1) \quad d\eta(X, Y) = g(X, \phi Y), \quad \eta(X) = g(X, \xi), \quad \phi^2 = -I + \eta \otimes \xi.$$

From these equations we have

$$(2.2) \quad \phi\xi = 0, \quad \eta\phi = 0, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

The manifold M equipped with the contact structure (ϕ, ξ, η, g) is called a contact metric manifold. A contact metric structure is said to be normal (Sasakian) if the almost complex structure J on $M \times R$ defined by, $J(X, f \frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$, f being a function on M^n , is integrable. A contact metric manifold is Sasakian if and only if

$$(2.3) \quad R(X, Y)\xi = \eta(Y)X - \eta(X)Y.$$

Every Sasakian manifold is K-contact, but the converse need not be true, except in dimension three ([12]). K-contact metric manifold are not too well known, because there is no such a simple expression for the curvature tensor as in the case of Sasakian manifold. For details we refer to ([2], [3], [15]).

Besides the above relations in K-contact manifold the following relations hold ([2], [3], [15]):

$$(2.4) \quad \nabla_X \xi = -\phi X.$$

$$(2.5) \quad g(R(\xi, X)Y, \xi) = \eta(R(\xi, X)Y) = g(X, Y) - \eta(X)\eta(Y).$$

$$(2.6) \quad R(\xi, X)\xi = -X + \eta(X)\xi.$$

$$(2.7) \quad S(X, \xi) = (n - 1)\eta(X).$$

$$(2.8) \quad (\nabla_X \phi) = R(\xi, X)Y.$$

for any vector fields X, Y .

Further since ξ is a Killing vector field, S and r remains invariant under it, i.e.,

$$(2.9) \quad L_\xi S = 0 \quad \text{and} \quad L_\xi r = 0,$$

where L denotes the Lie-derivation.

Again a K-contact manifold is called Einstein if the Ricci tensor S is of the form $S = \lambda g$, where λ is a constant and η -Einstein if the Ricci tensor S is of the form $S = ag + b\eta \otimes \eta$, where a, b are smooth functions on M . It is well known ([12]) that in a K-contact manifold a and b are constants. Also it is known that every manifold of constant curvature is an Einstein manifold. The converse is only true for dimension three. Again a compact Einstein K-contact manifold is Sasakian ([4]).

A Riemannian or semi-Riemannian manifold is said to be semi-symmetric ([9]) if $R(X, Y).R = 0$, where R is the Riemannian curvature tensor and $R(X, Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y . If a Riemannian manifold satisfies $R(X, Y).\tilde{C} = 0$, where \tilde{C} is the quasi-conformal curvature tensor, then the manifold is said to be quasi-conformally semi-symmetric manifold.

§3. Quasi-conformally flat K-contact manifolds

In 1967 Tanno [20] proved that a conformally flat K-contact manifold is of constant curvature $+1$ and Sasakian. In this section we consider quasi-conformally flat K-contact manifold. If a K-contact manifold $(M^n, \phi, \xi, \eta, g)$ is quasi-conformally flat, then from (1.1) we get

$$(3.1) \quad \begin{aligned} g(R(X, Y)Z, W) = & \frac{b}{a}[S(X, Z)g(Y, W) - S(Y, Z)g(X, W) \\ & + S(Y, W)g(X, Z) - S(X, W)g(Y, Z)] \\ & + \frac{r}{an}\left(\frac{a}{n-1} + 2b\right)[g(Y, Z)g(X, W) \\ & - g(X, Z)g(Y, W)]. \end{aligned}$$

Now putting $X = Z = \xi$ in (3.1) we obtain

$$(3.2) \quad \begin{aligned} g(R(\xi, Y)\xi, W) &= \frac{b}{a}[S(\xi, \xi)g(Y, W) \\ &\quad - S(Y, \xi)g(\xi, W) + S(Y, W)g(\xi, \xi) \\ &\quad - S(\xi, W)g(Y, \xi)] + \frac{r}{an}(\frac{a}{n-1} + 2b) \\ &\quad [g(Y, \xi)g(\xi, W) - g(\xi, \xi)g(Y, W)]. \end{aligned}$$

Now using (2.1), (2.2), (2.5) and (2.7) it follows from (3.2) that

$$(3.3) \quad S(Y, W) = Ag(Y, W) + B\eta(Y)\eta(W),$$

where A and B are given by

$$(3.4) \quad A = -\frac{a}{b} + \frac{r}{nb}(\frac{a}{n-1} + 2b) - (n-1).$$

$$(3.5) \quad B = \frac{a}{b} - \frac{r}{nb}(\frac{a}{n-1} + 2b) + 2(n-1).$$

It follows from (3.4) and (3.5) that $A + B = (n-1)$.

In view of the relation (3.3) we state the following:

Proposition 3.1. *A quasi-conformally flat K-contact manifold is an η -Einstein manifold.*

Putting $Y = W = e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, in (3.3) and taking summation over i , $1 \leq i \leq n$, we get

$$(3.6) \quad r = An + B.$$

Now with the help of (3.4) and (3.5) the equation (3.6) gives

$$(3.7) \quad [(n-2) + \frac{a}{b}][\frac{r}{n} + (1-n)] = 0,$$

Hence either

$$(3.8) \quad b = \frac{a}{2-n}$$

or

$$(3.9) \quad r = n(n-1).$$

If $b = \frac{a}{2-n}$, then putting this into (1.1) we get

$$(3.10) \quad \tilde{C}(X, Y)Z = aC(X, Y)Z,$$

where $C(X, Y)Z$ denotes the Weyl conformal curvature tensor. So the quasi-conformally flatness and conformally flatness are equivalent in this case. A conformally flat K-contact manifold (M^n, g) ($n \geq 5$) is of constant curvature ([20]). But a manifold of constant curvature is conformally flat. Hence a K-contact manifold is conformally flat if and only if it is locally isometric with a unit sphere $S^n(1)$. So in this case, M^n is locally isometric with a unit sphere.

If $r = n(n-1)$, then from (3.3), (3.4) and (3.5) we obtain

$$(3.11) \quad S(Y, Z) = (n-1)g(Y, Z).$$

This implies that M^n is an Einstein manifold. Putting (3.9) and (3.11) into (3.1) we obtain

$$(3.12) \quad R(X, Y, Z, W) = g(X, W)g(Y, Z) - g(X, Z)g(Y, W).$$

Thus M^n is of constant curvature $+1$. Hence it is locally isometric with a unit sphere $S^n(1)$. If M^n is locally isometric with a unit sphere $S^n(1)$, it is easy to see that M^n is quasi-conformally flat. This leads to the following theorem:

Theorem 3.1. *Let (M^n, g) ($n \geq 5$) be a K-contact manifold. Then M^n is quasi-conformally flat if and only if it is locally isometric with a unit sphere $S^n(1)$.*

It is known ([14]) that if a contact metric manifold M^n is of constant curvature c and dimension ≥ 5 , then $c = 1$ and the structure is Sasakian.

Since the manifold under consideration is of constant curvature $+1$, therefore by the above mentioned result we get the following:

Corollary 3.1. *A quasi-conformally flat K-contact manifold M^n ($n \geq 5$) is Sasakian.*

§4. K-contact manifold satisfying $\operatorname{div}\tilde{C} = 0$

This section deals with a K-contact Riemannian manifold satisfying

$$(4.1) \quad \operatorname{div}\tilde{C} = 0,$$

where div denotes the divergence of the quasi-conformal curvature tensor \tilde{C} .

Differentiating (1.1) covariantly along U , we obtain

$$(4.2) \quad \begin{aligned} (\nabla_U \tilde{C})(X, Y)Z &= a(\nabla_U R)(X, Y)Z \\ &+ b[(\nabla_U S)(Y, Z)X - (\nabla_U S)(X, Z)Y \\ &+ g(Y, Z)(\nabla_U Q)X - g(X, Z)(\nabla_U Q)Y] \\ &- \frac{[a + 2(n-1)b]dr(U)}{n(n-1)}[g(Y, Z)(X) \\ &- g(X, Z)(Y)]. \end{aligned}$$

Contraction of (4.2) yields

$$(4.3) \quad \begin{aligned} (\operatorname{div}\tilde{C})(X, Y)Z &= (a+b)[(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)] \\ &- \frac{a - (n-1)(n-2)b}{n(n-1)}[g(Y, Z)dr(X) \\ &- g(X, Z)dr(Y)]. \end{aligned}$$

From (4.1) and (4.3) it follows that

$$(4.4) \quad \begin{aligned} &(a+b)[(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)] \\ &= \frac{a - (n-1)(n-2)b}{n(n-1)}[g(Y, Z)dr(X) - g(X, Z)dr(Y)]. \end{aligned}$$

From (2.9) we get

$$(4.5) \quad (\nabla_\xi S)(Y, Z) = -S(\nabla_Y \xi, Z) - S(\nabla_Z \xi, Y).$$

Putting $X = \xi$ in (4.4) and then using (4.5) and $dr(\xi) = 0$ we get

$$(4.6) \quad \begin{aligned} (a+b)[S(\nabla_Y \xi, Z) + S(\nabla_Z \xi, Y) + (\nabla_Y S)(\xi, Z)] \\ = \frac{a - (n-1)(n-2)b}{n(n-1)}\eta(Z)dr(Y). \end{aligned}$$

From (2.7) we have

$$(4.7) \quad (\nabla_Y S)(\xi, Z) = (n-1)(\nabla_Y \eta)(Z) - S(\nabla_Y \xi, Z).$$

Again using the relation $(\nabla_Y \eta)(Z) = g(\nabla_Y \xi, Z)$ in (4.7) we obtain

$$(4.8) \quad (\nabla_Y S)(\xi, Z) = (n-1)g(\nabla_Y \xi, Z) - S(\nabla_Y \xi, Z).$$

Using (4.8) in (4.6) we get

$$(4.9) \quad \begin{aligned} & (a+b)[(n-1)g(\nabla_Y \xi, Z) + S(\nabla_Z \xi, Y)] \\ &= \frac{a - (n-1)(n-2)b}{n(n-1)} \eta(Z) dr(Y). \end{aligned}$$

In view of (2.4) we obtain from (4.9)

$$(4.10) \quad \begin{aligned} & -(a+b)[(n-1)g(\phi Y, Z) + S(\phi Z, Y)] \\ &= \frac{a - (n-1)(n-2)b}{n(n-1)} \eta(Z) dr(Y). \end{aligned}$$

Replacing Z by ϕZ in (4.10) and using (2.1) we get

$$(4.11) \quad (a+b)[S(Y, Z) - (n-1)g(Y, Z)] = 0,$$

which implies

$$(4.12) \quad S(Y, Z) = (n-1)g(Y, Z),$$

provided $a+b \neq 0$. Hence (4.12) follows that

$$(4.13) \quad QY = (n-1)Y.$$

Hence in view of (4.12) and (4.13) we get from (1.1)

$$(4.14) \quad \tilde{C}(X, Y)Z = a[R(X, Y)Z + g(X, Z)Y - g(Y, Z)X].$$

From (4.12) we get the following:

Theorem 4.1. *A K-contact manifold with divergence free quasi-conformal curvature tensor is an Einstein manifold provided $a+b \neq 0$.*

Since a compact K-contact Einstein manifold is Sasakian ([4]), hence we obtain the following:

Corollary 4.1. *A compact K-contact manifold with divergence free quasi-conformal curvature tensor is Sasakian.*

If a K-contact manifold (M^n, g) ($n \geq 5$) is quasi-conformally symmetric, then it satisfies $\text{div}\tilde{C} = 0$. Hence the relation (4.14) holds from which it follows that the manifold is locally symmetric. Again it is known ([20]) that a locally symmetric K-contact manifold is of constant curvature +1 and Sasakian. Hence we state the following:

Corollary 4.2. *A quasi-conformally symmetric K-contact manifold (M^n, g) ($n \geq 5$) is of constant curvature +1 and Sasakian.*

§5. K-contact manifold satisfying $R(X, Y).\tilde{C} = 0$

To solve this problem we consider following two cases:

Case i) The manifold is Einstein.

Case ii) The manifold is η -Einstein.

Case i) In this case we have,

$$(5.1) \quad S(X, Y) = \lambda g(X, Y),$$

where λ is a constant. Putting $X = Y = \xi$ in (5.1) we get by virtue of (2.1), (2.2) and (2.7) that $\lambda = n - 1$. Hence (5.1) reduces to

$$(5.2) \quad S(X, Y) = (n - 1)g(X, Y),$$

which yields

$$(5.3) \quad QX = (n - 1)X$$

and

$$(5.4) \quad r = n(n - 1).$$

Now from (1.1), using (5.2), (5.3) and (5.4), we get

$$(5.5) \quad \begin{aligned} \tilde{C}(X, Y)Z &= aR(X, Y)Z + [2b(n - 1) - \frac{ar}{n(n - 1)} \\ &\quad - \frac{2br}{n}][g(Y, Z)X - g(X, Z)Y]. \end{aligned}$$

Therefore we get

$$(5.6) \quad R.\tilde{C} = aR.R.$$

From (5.6) it follows that on a K-contact Einstein manifold the quasi-conformally semi-symmetry and semi-symmetry are equivalent, since by assumption $a \neq 0$.

It is known ([21]) that a semi-symmetric K-contact manifold is of constant curvature +1 and Sasakian. Hence an Einstein quasi-conformally semi-symmetric K-contact manifold is of constant curvature +1 and Sasakian.

Case ii) In this case we have,

$$(5.7) \quad S(X, Y) = \alpha g(X, Y) + \beta \eta(X)\eta(Y),$$

where α and β are scalars. Putting $X = Y = \xi$ in (5.7) we get

$$(5.8) \quad (n - 1) = \alpha + \beta.$$

Let $\{e_i\}$ ($i = 1, 2, \dots, n$) be the orthonormal basis of the tangent space at each point of the manifold M . Then putting e_i in the place of X and Y of (5.7) and summing up over 1 to n we get

$$(5.9) \quad r = \alpha n + \beta.$$

Solving (5.8) and (5.9) we have

$$(5.10) \quad \alpha = \frac{r}{n-1} - 1, \quad \beta = n - \frac{r}{n-1}.$$

Again (5.7) yields

$$(5.11) \quad QX = \alpha X + \beta \eta(X)\xi.$$

Then using (5.7), (5.10) and (5.11) in (1.1) we get

$$(5.12) \quad \begin{aligned} \tilde{C}(X, Y)Z &= aR(X, Y)Z \\ &+ [2b(\frac{r}{n-1} - 1) - \frac{r}{n}(\frac{a}{n-1} + 2b)] \times \\ &\quad [g(Y, Z)X - g(X, Z)Y] \\ &+ b(n - \frac{r}{n-1})[g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi] \\ &+ b(n - \frac{r}{n-1})[\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned}$$

Then from (5.12) we have

$$\begin{aligned}
 (5.13) \quad \check{\check{C}}(X, Y, Z, W) = & a\check{R}(X, Y, Z, W) \\
 & + [2b(\frac{r}{n-1} - 1) - \frac{r}{n}(\frac{a}{n-1} + 2b)] \times \\
 & [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\
 & + b(n - \frac{r}{n-1})[g(Y, Z)\eta(X)\eta(W) \\
 & - g(X, Z)\eta(Y)\eta(W)] \\
 & + b(n - \frac{r}{n-1})[\eta(Y)\eta(Z)g(X, W) \\
 & - \eta(X)\eta(Z)g(Y, W)],
 \end{aligned}$$

where

$$\check{\check{C}}(X, Y, Z, W) = g(\check{C}(X, Y)Z, W) \text{ and } \check{R}(X, Y, Z, W) = g(R(X, Y)Z, W).$$

Since $r, g, \eta(X), \eta(Y)$ and $\eta(Z)$ all are scalars, then (5.13) yields

$$(5.14) \quad R(X, Y).\check{\check{C}} = aR(X, Y).\check{R}.$$

Thus from (5.14) we have quasi-conformally semi-symmetry and semi-symmetry are equivalent since by assumption $a \neq 0$.

It is well known ([21]) that a semi-symmetric K-contact manifold is of constant curvature +1 and Sasakian.

Therefore from the above discussions we can state the following theorem:

Theorem 5.1. *A quasi-conformally semi-symmetric K-contact Einstein (or η -Einstein) manifold is a manifold of constant curvature +1 and Sasakian.*

Since a manifold of constant curvature +1 is locally isometric with a unit sphere, hence we have:

Corollary 5.1. *A quasi-conformally semi-symmetric K-contact Einstein (or η -Einstein) manifold is locally isometric with a unit sphere.*

§6. ξ -quasi-conformally flat K-contact manifold

ξ -conformally flat K-contact manifolds have been studied by Zhen, Cabrerizo, L. M. Fernandez and M. Fernandez [23]. Here we study ξ -quasi-conformally

flat K-contact manifold.

Definition 6.1. A K-contact manifold is said to be ξ -quasi-conformally flat ([23]) if $\tilde{C}(X, Y)\xi = 0$.

Let us assume that the manifold M^n is ξ -quasi-conformally flat. Then using $\tilde{C}(X, Y)\xi = 0$ in (1.1) we get

$$(6.1) \quad \begin{aligned} aR(X, Y)\xi &+ b[(n-1)\eta(Y)X - (n-1)\eta(X)Y + \eta(Y)QX \\ &- \eta(X)QY] - \frac{r}{n}(\frac{a}{n-1} + 2b)[\eta(Y)X - \eta(X)Y] \\ &= 0. \end{aligned}$$

Putting $X = \xi$ in (6.1) and using (2.6) and $\eta(\xi) = 1$ we get

$$(6.2) \quad \begin{aligned} a(-Y + \eta(Y)\xi) &+ b[(n-1)(\eta(Y)\xi - Y) + \eta(Y)Q\xi - QY] \\ &- \frac{r}{n}(\frac{a}{n-1} + 2b)[\eta(Y)\xi - Y] = 0. \end{aligned}$$

i.e,

$$(6.3) \quad S(Y, W) = Ag(Y, W) + B\eta(Y)\eta(W),$$

where A and B are given by

$$(6.4) \quad A = -\frac{a}{b} + \frac{r}{nb}(\frac{a}{n-1} + 2b) - (n-1)$$

and

$$(6.5) \quad B = \frac{a}{b} - \frac{r}{nb}(\frac{a}{n-1} + 2b) + 2(n-1).$$

In view of (6.3) we state the following:

Theorem 6.1. A ξ -quasi-conformally flat K-contact manifold is an η -Einstein manifold.

Let us assume that there exist two functions L and M on M^n such that

$$(6.6) \quad (\nabla_X Q)Y - (\nabla_Y Q)X = LX + MY,$$

for $X, Y \in T(M)$.

From (6.3) we have

$$(6.7) \quad QX = AX + B\eta(X)\xi,$$

where A and B are given by (6.4) and (6.5) respectively. Thus we have

$$(6.8) \quad \begin{aligned} (\nabla_X Q)Y - (\nabla_Y Q)X &= (XA)Y - (YA)X + (XB)\eta(Y)\xi \\ &\quad - (YB)\eta(X)\xi - B\eta(Y)\phi X \\ &\quad + B\eta(X)\phi Y - 2Bg(\phi X, Y)\xi. \end{aligned}$$

Replacing X by ϕX and Y by ϕY in (6.8), we get

$$(6.9) \quad \begin{aligned} (\nabla_{\phi X} Q)\phi Y - (\nabla_{\phi Y} Q)\phi X &= (\phi X A)\phi Y - (\phi Y A)\phi X \\ &\quad - 2Bg(\phi^2 X, \phi Y)\xi. \end{aligned}$$

From (6.6) and (6.9) we obtain

$$(L + (\phi Y A))\phi X + (M - (\phi X A))\phi Y = -2Bg(\phi^2 X, \phi Y)\xi,$$

which shows that

$$(6.10) \quad -2Bg(\phi^2 X, \phi Y) = 0.$$

Replacing X by ϕY in (6.10) we have

$$(6.11) \quad 2Bg(\phi Y, \phi Y) = 0.$$

Hence from (6.11), it follows that $B = 0$. Therefore from (6.7), we get $QX = AX$. Then from (2.7) we obtain $QX = (n - 1)X$. Therefore we have the following:

Corollary 6.1. *Let M^n be a ξ -quasi-conformally flat K-contact manifold. If there exist functions L and M on M^n such that*

$$(\nabla_X Q)Y - (\nabla_Y Q)X = LX + MY,$$

for $X, Y \in T(M)$, then

$$(6.12) \quad QX = (n - 1)X.$$

From Corollary 6.1 we have the following applications:

Corollary 6.2. *A quasi-conformally flat K-contact manifold is of constant curvature +1 and Sasakian.*

Proof: Let us suppose that a K-contact manifold is quasi-conformally flat. Then from (4.4) it follows that

$$(6.13) \quad (\nabla_X Q)(Y) - (\nabla_Y Q)(X) = \frac{[a - (n-1)(n-2)b]}{(a+b)n(n-1)} [Ydr(X) - Xdr(Y)].$$

Hence by the above Corollary 6.1 we obtain the manifold is an Einstein manifold. Using (6.12) and (6.13) in (3.1) we obtain

$$(6.14) \quad R(X, Y)Z = g(Y, Z)X - g(X, Z)Y.$$

Therefore the quasi-conformally flat K-contact manifold is of constant curvature +1 and Sasakian. This proves the Corollary.

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